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Comparison between samples with constant mass and samples with constant fragment population size

(and calculations of their sampling variances)

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Dominique is the founder of Agoratek International Consultants Inc., a consulting practice that includes sampling services in its palette, along with reconciliations and all related disciplines. An active and passionate researcher in the theory of sampling, he is a recipient of Pierre Gy's gold medal for excellence in promoting and teaching the Theory of Sampling (WCSB3). Francis is the founder of Francis Pitard Sampling Consultants (FPSC) and has provided professional sampling consultation services and educational programs to many of the world's leading companies. *Pierre Gy's Sampling Theory and Sampling Practice*, 2nd Edition, CRC Press (1993) is one of his many publications. He is a recipient of Pierre Gy's gold medal for excellence in promoting and teaching the Theory of Sampling (WCSB4).

A short introduction from the "Revue de l'Industrie Minerale"

In the 1960s the "Revue de l'Industrie Minerale" took the delicate task to publish several special editions in its journal to expose the Theory of Sampling suggested by Pierre Gy. They wrote:

"In these documents Pierre Gy suggests an equi-probabilistic Theory of Sampling based on samples with a constant number of fragments; in other words, considering samples that are all made of the same number of fragments. Nevertheless, results from his theoretical analysis lead to the justification that formulas that are suggested in practice are applicable to samples with constant mass or constant volume as well. It should be underlined that similar American studies seem to support Pierre Gy's opinion."

G. Matheron, through a careful review of Pierre Gy's work, has demonstrated, by means of a rigorous mathematical analysis that both samples with a constant number of fragments and samples with a constant mass lead to a dispersion of possible grades for the lot to be sampled that have similar variances.

The mathematical level of this study may prove to be difficult to many readers. However, the importance of the argument is critical for the validity of the Theory of Sampling; it is an argument that has been approached by many authors over the years leading to frustration and failure. As a consequence, Pierre Gy's theory generated passionate controversies.

The "Revue de l'Industrie Minerale" is proud to bring to this important discussion the contribution of an authority as famous as G. Matheron."

This document was published 55 years ago and still endures the challenges of time. During that long period of time it became clear that the fact that such an important document was written in French was a huge handicap, especially for reaching the Anglo-Saxon audience effectively. We hope the present translation made by two recipients of Pierre Gy's Gold Medal for excellence in teaching the Theory of Sampling will help to fill that gap. Complementary explanations are also inserted where appropriate, so the reader can progress in a more friendly way, and better appreciate the subtle foundations of the Theory of Sampling.

Abstract

In his essay "l'echantillonnage des minerais en vrac" that could be translated as "sampling of particulate ore" published in 1967 in France by the *Revue de l'Industrie Minerale*, Pierre Gy suggests a calculation of the variance associated with samples with a constant number of fragments. In practice, samples with a constant mass are instead collected, which may seem at first like a contradiction. In this mathematical development it is clearly demonstrated that these two kinds of samples lead to variances that are similar within well-established mathematical limits.

[Translators' Note (T.N.): To make the translation easier to read, the structure and the logical articulation of Matheron's long and tedious paper need to be understood first:

1. *The introduction first establishes the difference between "sampling in number" and "sampling in mass", to ready the mathematical background.*
2. ***Sampling in Number** is then studied.*
   - *This first calls for specific developments aimed at calculating first order approximations to* $E(1/Y)$, $E(1/Y^k)$ *and, for Gy's formula,* $E(X^k/Y^k)$ *for a random variable* Y. *There are no ready-made formulas in statistics, and no exact formulas for this task.*
   - *A rather tedious demonstration using the Laplace transform indeed shows that in the case of* $E(1/Y)$, *for any random variable* Y *that can be interpreted as the mean of "*n*" independent, identically distributed (i.d.d.) random variables, even a divergent serial development can be actually used as a limited development near the value of "*n*" considered. The result is then generalized to* $E(1/Y^k)$.
   - *Having set up these mathematical tools, then Gy's formula is established for sampling in number and the result is formally identical to Gy's findings, thus validating it.*
3. ***Sampling in Mass** is then tackled:*
   - *The necessary approximations are again established for* $E(N)$ *and* $E(N^2)$, N *being the number of fragments (now a random variable) in the sample of a given mass* p.
   - *Then approximations are also needed for* $E(X;p)$ *and* $E(X^2;p)$, *i.e. the mathematical expectations of the sample metal*

quantity X at a given mass p, and of its square. This time, the Fourier transform helps establish, like before, usable limited developments.

- *The same formula as for sampling in number is finally reached in the case of a given mass, and the paper concludes that Gy's formula is formally and fully validated for both types of sampling.]*

1. Matheron's introduction

In his fundamental document published in 1967 dedicated to "l'echantillonnage des minerais en vrac" (i.e. the sampling of bulk ores) Pierre Gy demonstrated that the variance associated with a sample made of a given number of fragments n (i.e., sample made of a preset number n of elementary fragments) follows an asymptotic tendency when the number n is large and when the sampling mode that is used is equi-probabilistic (i.e., correct, which means all possible samples with n fragments have the same probability of being selected by the sampling tool). The theoretical path followed by Pierre Gy has been the object of severe criticism over the years that can be summarized by the two following arguments:

1. As far as the calculation of the variance is concerned, the validity of some of Gy's developments has been contested.
2. Furthermore, in daily practice, the collection of samples with a constant number of fragments n predetermined in advance is never done that way, but rather samples with a predetermined mass or volume are indeed collected; the number of fragments in these real samples is then necessarily unknown. Then, consequently, it is apparently justified to doubt that conclusions reached for samples with a constant number of fragments are also valid for samples with a constant mass or a constant volume.

The objective of this study is to carefully investigate these objections and demonstrate the full legitimacy of Pierre Gy's results *[T.N.: in other words it is a corner stone to confirm the legitimacy of the Theory of Sampling]*. From a mathematical standpoint it is relatively easy to demonstrate the legitimacy of a theory based on samples with a constant number of fragments. However, to legitimately transfer these results to a theory based on samples of constant mass or constant volume requires the use of far more difficult mathematical tools. In a way this explains why Pierre Gy in his search for a pragmatic tool chose to use the simplest approach. For the sake of simplicity and to avoid unnecessary mathematical developments the assumption is made that the original number of fragments in the lot to be sampled is practically infinite, which is most of the time almost exactly the case. To get straight to the point the two following hypotheses are made:

The mass $\bar{\omega}$ of one ore fragment and its metal content q can be considered as two random variables that are not independent. Furthermore, to simplify notations, the assumption is made that their distribution function $F(q \cdot \bar{\omega})$ carries a probability density function $f(q,\bar{\omega})$.

1. The sample collection mode is such that the selected sample can be considered as the reunion of fragments following the same probability law $f(q,\bar{\omega})$ (i.e., collected one by one at random and making sure they are independent from one another). In other words:
   a. For a sample with a given number of fragments n, its mass and its metal content can be written as follows:

$$X_n = \sum_{i=1}^{n} q_i \qquad [1]$$

$$Y_n = \sum_{i=1}^{n} \bar{\omega}_i \qquad [2]$$

It is understood that each q_i is independent of q_j and $\bar{\omega}_j$ for $j \neq i$ (but it is also understood that q_i and $\bar{\omega}_i$ are not independent), and each pair $(q_i,\bar{\omega}_i)$ follows the same probability law $f(q,\bar{\omega})$.

   b. If, on the contrary, we consider a sample of a given mass p defined by the following condition:

$$\sum_{i=1}^{N} \bar{\omega}_i < p \quad \text{and} \quad \sum_{i=1}^{N+1} \bar{\omega}_i \geq p \qquad [3]$$

its number of fragments N appears to be a random variable. The mass Y_n and the quantity of metal X_n of this sample are then defined as sums of a random number N of variables $\bar{\omega}_i$ or q_i.

In a first part we will calculate the mathematical expectation and the variance of the metal concentration X_n/Y_n of the sample carrying a number of fragments n when n is a large number. In the second part the same calculations are repeated for the metal concentration X_N/p of a sample of a given mass p, when p is a large number. It is intended to demonstrate that their variances for both cases are asymptotically equivalent when p is large.

2. Case of a sample with a constant number of fragments

Assuming the number of fragments in the sample is n let us introduce:

$$X = \frac{1}{n} X_n = \frac{1}{n} \sum_{i=1}^{n} q_i \qquad [4]$$

$$Y = \frac{1}{n} Y_n = \frac{1}{n} \sum_{i=1}^{n} \bar{\omega}_i \qquad [5]$$

When n is large, variances of X and Y are in $1/n$, and the centered moments of superior orders are at least in $1/n^2$. The metal concentration X/Y of the sample appears then like the ratio of two random variables having very small variances. To calculate its mathematical expectation and its variance, it is convenient to write:

$$Y = m_y + \varepsilon \quad \text{with} \quad m_y = E(Y) \qquad [6]$$

and then getting started from the following formal development in series:

$$\frac{X}{Y} = \frac{X}{m_y}\left[1 - \frac{\varepsilon}{m_y} + \frac{\varepsilon^2}{(m_y)^2} + \ldots + (-1^n)\frac{\varepsilon^n}{(m_y)^n} + \ldots\right] \qquad [7]$$

which can be used if $|\varepsilon/m_y| < 1$.

If there is a probability of "one" that the inequality $|\varepsilon/m_y| < 1$ be verified, it is possible then to take the mathematical expectation one term at a time, and then deduce from a complete serial development what the expression of the average and the variance of X/Y should be. Of course, if the inequality $|\varepsilon/m_y| < 1$ is not verified with a probability of "one", this mode of calculation would no longer be valid, plus the serial developments that could be obtained would generally diverge anyway. Nevertheless, as a limited development (as opposed to a formal serial development) the results obtained by this process would conserve their validity.

2.1 Mathematical expectation $E(1/Y)$

To prove it without useless mathematical developments, let's focus only on the mathematical expectation $E(1/Y)$ of a variable Y characterized by a density $f(y)$. In this case indeed:

$$E\left(\frac{1}{Y}\right)=\int_{-\infty}^{+\infty}\frac{1}{y}f(y)dy \qquad [8]$$

If $f(0) \neq 0$, this integral is divergent and therefore $1/Y$ cannot have a mathematical expectation: for example, if Y is a normal variable, its inverse can never have a mathematical expectation. Therefore it is critically important not to assume that the law of $f(y)$ is close to a Gaussian law. In fact, Y representing a mass, $f(y)$ is different of *0* only for $y \geq 0$, and the integral [8] will exist provided the density $f(y)$ is of a very small order $\varepsilon > 0$ for $y = 0$. It is easy to demonstrate that this condition is always satisfied in the case where Y is the sum of at least two independent variables that themselves follow continuous laws. In the problem that is investigated in this study $E(1/Y)$ therefore always exists *(T.N.: because of equation [5])*.

To evaluate $E(1/Y)$ it is convenient to introduce the Laplace transform of the law $f(y)$:

$$\Phi(\lambda)=\int_0^{\infty}e^{-\lambda y}f(y)dy \qquad [9]$$

As a matter of fact $\Phi(\lambda)$ always exists for $\lambda \geq 0$ and the following integral as well:

$$\int_0^{\mu}\Phi(\lambda)d\lambda=\int_0^{\infty}\frac{1-e^{-\mu y}}{y}f(y)dy \qquad [10]$$

If μ tends toward the infinite in relation [10] it can be noticed that the mathematical expectation of $1/Y$ exists at the same time as the integral

$\int_0^{\mu}\Phi(\lambda)d\lambda$, and that:

$$E\left(\frac{1}{Y}\right)=\int_0^{\infty}\Phi(\lambda)d\lambda \qquad [11]$$

(T.N.: i.e. the integral of the Laplace transform on [0,∞].)

Let's call m the mean, σ^2 the variance and μ_n the centered moment of order n of the variable Y (that we will assume *to* exist); let's also call $\Phi_c(\lambda)$ the Laplace transform of the law of the centered variable $Y-m$:

$$\Phi_c(\lambda)=E\left[e^{-\lambda(Y-m)}\right]=e^{\lambda m}\Phi(\lambda) \qquad [12]$$

Then also:

$$E\left(\frac{1}{Y}\right)=\int_0^{\infty}e^{-\lambda m}\Phi_c(\lambda)\ d\lambda \qquad [13]$$

Now, in some cases $\Phi_c(\lambda)$ can be developed into a formal series of the following form:

$$\Phi_c(\lambda)=1+\frac{\lambda^2\sigma^2}{2!}+\ldots+(-1)^k\frac{\lambda^k\mu_k}{k!}+\ldots \qquad [14]$$

Taking this expression into [13] and if it is integrated term by term, the following expression is obtained *(T.N.: after quite some calculus):*

$$E\left(\frac{1}{Y}\right)=\frac{1}{m}+\frac{\sigma^2}{m^3}+\ldots+(-1)^k\frac{\mu_k}{m^{k+1}}+\ldots \qquad [15]$$

In other words, it is exactly the result that should be expected from the development [7] *[T.N.: with* X = *1 and* Y – m = *ε]*.

However:

- It is possible that the series [14] may not be convergent; this is the case when Y is, for instance, a lognormal variable.
- It is also possible that even if the series [14] is convergent, it may not be uniformly convergent, making the term by term integration invalid; it is what happens, for instance, when f is a gamma law:

$$f(y)=\frac{b^{\alpha}}{\Gamma(\alpha)}y^{\alpha-1}e^{-by} \qquad [16]$$

For $\alpha > 1$, $E(1/Y)$ exists and [14] converges. However, the convergence is not uniform and it is easily shown that the formal mathematical development written in [15] is diverging (μ_k/m^k tends toward an infinite value with k).

Therefore, in general, it is not possible to use the full development written in [15] as a formal series.

However, as we are going to see, it is always possible to use it as a limited development within the domain of variations that is of interest to us, provided we can show the rest of the development behaves as a negligible remainder in that domain.

As a matter of fact let's write:

$$\Phi_c(\lambda)=1+\lambda^2\frac{\sigma^2}{2!}+\ldots+\frac{(-1)^k}{k!}\lambda^k\mu_k+R_k(\lambda) \qquad [17]$$

The remainder $R_k(\lambda)$ of this development is:

$$R_k(\lambda)=\int_0^{\infty}\left[e^{-\lambda(x-m)}-1-\lambda^2\frac{(x-m)^2}{2!}-\ldots-\frac{(-1)^k\lambda^k}{k!}(x-m)^k\right]f(x)dx \qquad [18]$$

By taking [17] into [13] the following expression is obtained:

$$E\left(\frac{1}{Y}\right)=\frac{1}{m}+\frac{\sigma^2}{m^3}+\ldots+(-1)^k\frac{\mu_k}{m^{k+1}}+R'_k \qquad [19]$$

The remainder R'_k of this development is:

$$R'_k=\int_0^{\infty}e^{-\lambda m}R_k(\lambda)d\lambda=\int_0^{\infty}\frac{1}{x}\left(1-\frac{x}{m}\right)^{k+1}f(x)dx \qquad [20]$$

To find an upper bound for this remainder let's take a number $\alpha > 1$ (that we will soon define) and let's write:

$$R'_k=\int_0^{\frac{m}{\alpha}}\frac{1}{x}\left(1-\frac{x}{m}\right)^{k+1}f(x)dx+\int_{\frac{m}{\alpha}}^{\infty}\frac{1}{x}\left(1-\frac{x}{m}\right)^{k+1}f(x)dx \qquad [21]$$

For $x \geq (m/\alpha)$ we have:

$$\left|\frac{1}{x}\left(1-\frac{x}{m}\right)^{k+1}\right|\leq\frac{\alpha}{m^{k+2}}|m-x|^{k+1} \qquad [22]$$

so that an upper bound for the second integral is:

$$\frac{\alpha}{m^{k+2}}E\left[\ |Y-m|^{k+1}\right] \qquad [23]$$

But, if Y can be written like

$\frac{1}{n}\sum_{i=1}^{n}Y_i$

the centered absolute moment

$E\left[\,|Y-m|^{k+1}\right]$

tends toward *0* when *n* tends towards infinity (generally it is an infinitesimally small value of order $h+1<k+1$ in $1/n$).

What is left to do is finding an upper bound for the first integral of remainder R'_k.

If the density of Y_i has a number *B* as upper bound, the density of *Y* verifies:

$$f(x)\le (nB)^n\frac{x^{n-1}}{(n-1)!} \qquad [24]$$

[T.N.: this upper bound can be difficult to establish. First, one must show the density function of the sum of the n *variables* X_i *has*

$$\left\{B^n\frac{x^{n-1}}{(n-1)!}\right\}$$

as upper bound. This is obtained by recurrence, remembering the density function g(t) *of the sum of two independent random variables with positive values is the convolution product of their densities (summed between 0 and* t*). The upper bound for the average of the* X_i *is then only derived, remembering the density function* h(t) *of variable "*X/n*" can be derived from the density* f(t) *of* X *as* h(t) = n f(nt)*.]*

Then, the following expression is obtained:

$$\int_0^{\frac{m}{\alpha}}\frac{1}{x}\left(1-\frac{x}{m}\right)^{k+1}f(x)\,dx\le (nB)^n\int_0^{\frac{m}{\alpha}}\frac{x^{n-2}}{(n-1)!}\,dx=\frac{(nB)^n}{(n-1)(n-1)!}\left(\frac{m}{\alpha}\right)^{n-1} \qquad [25]$$

If *n* is sufficiently large the Stirling formula can be used to replace factorials in [25] by a term such as

$$(Be)^n\left(\frac{m}{\alpha}\right)^{n-1}$$

that tends exponentially toward *0* when $n\to\infty$ **as long as the selected value for** α is superior to *Bme* (e.g., $\alpha = 3Bm$).

Therefore, finally, the last part R'_k is an infinitesimally small number in $1/n$ in the order of

$$\frac{1}{m^{k+2}}E\left[|Y-m|^{k+1}\right]$$

(an order generally smaller than $k+1$). It is not important if R'_k tends toward 0 or not for $k\to\infty$. If, for any given value of *k,* it is possible to verify that R'_k is in the order of $h+1$ in $1/n$, it is then possible in the approximation of order *h* (generally $< k$) to utilize the development [19] and stop at the term in μ_k.

2.2 Mathematical expectation $E(1/Y^k)$

Similarly it is possible to demonstrate that the mathematical expectation $E(1/Y^k)$ exists at the same time as the integral

$$\int_0^\infty\frac{\lambda^{k-1}}{(k-1)!}\Phi(\lambda)\,d\lambda$$

and then:

$$E\left[\frac{1}{Y^k}\right]=\int_0^\infty\frac{\lambda^{k-1}}{(k-1)!}\Phi(\lambda)\,d\lambda \qquad [26]$$

It is then sufficient to replace $\Phi(\lambda)$ with $e^{-\lambda m}\Phi_c(\lambda)$ to obtain, as above [see (13)], the formal development of $E(1/Y^k)$ which is most of the time divergent; however, it is possible to use it as a limited development. For example, for $k = 2$, it becomes:

$$E\left(\frac{1}{Y^2}\right)=\frac{1}{m^2}\left(1+\frac{3\sigma^2}{m^2}-4\frac{\mu_3}{m^3}+\ldots\right) \qquad [27]$$

from which (by subtracting the square of $E(1/Y)$ in (19), it is possible to obtain the principal part of the variance of $1/Y$:

$$D^2\left(\frac{1}{Y}\right)=\frac{\sigma^2}{m^4}+\ldots \qquad [28]$$

2.3 Establishing Gy's formula

If *X* and *Y* are two random variables of law *f(x,y)* we can start by introducing the Laplace transform:

$$\Phi(\lambda,\mu)=\int_0^\infty\int_0^\infty e^{-\lambda x-\mu y}f(x,y)\,dxdy \qquad [29]$$

And, as above, the following relation is obtained:

$$E\left[\frac{X^k}{Y^k}\right]=(-1)^k\int_0^\infty\frac{\mu^{k-1}}{(k-1)!}\cdot\frac{\partial^k\Phi(\lambda,\mu)}{\partial\lambda^k}d\mu\;\Big|_{\lambda=0} \qquad [30]$$

[T.N.: To see it, one first calculates:

$$\frac{\partial^k\Phi(\lambda,\mu)}{\partial\lambda^k}d\mu\;\Big|_{\lambda=0}=\int_0^\infty\int_0^\infty(-1)^k\,x^k e^{-\mu y}f(x,y)\,dxdy$$

So that the right-hand side of [30] is:

$$I=\int_0^\infty\frac{\mu^{k-1}}{(k-1)!}\cdot d\mu\int_0^\infty\int_0^\infty x^k e^{-\mu y}f(x,y)\,dxdy$$

Noting that:

$$E\left[\frac{1}{Y^k}\right]=\int_0^\infty\frac{f(y)}{y^k}\,dy=\int_0^\infty\frac{\mu^{k-1}}{(k-1)!}\cdot d\mu\int_0^\infty e^{-\mu y}f(y)\,dy$$
$$=\int_0^\infty\int_0^\infty\int_0^\infty\frac{\mu^{k-1}}{(k-1)!}e^{-\mu y}f(x,y)\,dx\,dy\,d\mu$$

One can see that:

$$I=\int_0^\infty\int_0^\infty\frac{x^k}{y^k}f(x,y)\,dxdy=E\left[\frac{X^k}{Y^k}\right]$$

(Q.E.D.)]

Then, calling Φ_c the transform of the centered variables law:

$$\Phi_c(\lambda,\mu)=E\left[e^{-\lambda(X-m_x)-\mu(Y-m_y)}\right]=1+\frac{\lambda^2\sigma_x^2+2\lambda\mu\sigma_{xy}+\mu^2\sigma_y^2}{2!}+\ldots \qquad [31]$$

Suffice substituting Φ for $e^{-\lambda m_x-\mu m_y}\Phi_c$ in equation [30], as before, to obtain some formal developments, generally divergent, that can be used as limited developments. So that, from:

$$-\frac{\partial}{\partial\lambda}\Phi(\lambda,\mu)\big|_{\lambda=0}=\left[m_x\left(1+\frac{\mu^2}{2}\sigma_y^2+\ldots\right)-\mu\sigma_{xy}+\ldots\right]e^{-\mu m_y} \qquad [32]$$

$$\frac{\partial^2}{\partial\lambda^2}\Phi(\lambda,\mu)\big|_{\lambda=0}=\left[m_x^2\left(1+\frac{\mu^2}{2}\sigma_y^2\right)-2\mu m_x\sigma_{xy}+\sigma_x^2+\ldots\right]e^{-\mu m_y} \qquad [33]$$

one obtains without difficulty the first terms of the developments of $E(X/Y)$ and $E(X^2/Y^2)$:

$$E\left(\frac{X}{Y}\right)=\frac{m_x}{m_y}\left[1+\frac{\sigma_y^2}{(m_y)^2}-\frac{\sigma_{xy}}{m_x m_y}+\ldots\right] \qquad [34]$$

$$E\left(\frac{X^2}{Y^2}\right)=\left(\frac{m_x}{m_y}\right)^2\left[1+3\frac{\sigma_y^2}{(m_y)^2}-4\frac{\sigma_{xy}}{m_x m_y}+\frac{\sigma_x^2}{(m_x)^2}+\ldots\right] \quad [35]$$

as well as the principal part of the variance of X/Y:

$$D^2\left(\frac{X}{Y}\right)=\left(\frac{m_x}{m_y}\right)^2\left[\frac{\sigma_x^2}{(m_x)^2}-\frac{2\sigma_{xy}}{m_x m_y}+\frac{\sigma_y^2}{(m_y)^2}+\ldots\right] \quad [36]$$

which can also be written:

$$D^2\left(\frac{X}{Y}\right)=\left(\frac{m_x}{m_y}\right)^2 E\left[\left(\frac{X}{m_x}-\frac{Y}{m_y}\right)^2\right] \quad [37]$$

Coming back to the original notations shown in [1] and [2], it is easy to see that the metal content $X/Y = X_n/Y_n$ of the sample with n fragments has a mathematical expectation and a variance that can be written as follows [when ignoring the terms in $1/n^2$ and letting $x_0 = E(q)$ and $y_0 = E(\bar{\omega})$]

$$E\left(\frac{X}{Y}\right)=\frac{x_0}{y_0}+\frac{1}{n}\cdot\frac{x_0}{y_0}E\left[\frac{\bar{\omega}}{y_0}\left(\frac{\bar{\omega}}{y_0}-\frac{q}{x_0}\right)\right] \quad [38]$$

$$D^2\left(\frac{X}{Y}\right)=\frac{1}{n}\left(\frac{x_0}{y_0}\right)^2 E\left[\left(\frac{q}{x_0}-\frac{\bar{\omega}}{y_0}\right)^2\right] \quad [39]$$

These are indeed the formulas obtained by Pierre Gy and in particular formulas obtained in Chapter IV of his January 15, 1967 publication. The validity of these formulas is therefore no longer questionable. It can be seen that the mathematical expectation $E(X/Y)$ of the metal content of the sample made of n fragments (for all the possible choices of the sample in the lot) does not exactly coincide with the real mean of the lot, which is x_0/y_0, but the difference is extremely small and in $1/n$. Therefore there is indeed always a small bias. The variance is, as expected, as the inverse of size n.

3. Case of a sample with constant mass

In daily practice it is clear that collected samples have either a constant mass, or volume, pre-selected in advance, rather than a constant number n of fragments. Then it is not obvious that the sampling variances for these two sampling modes are the same. Because the variable

$$\frac{1}{n}\sum_{i=1}^{n}\bar{\omega}$$

almost surely converges toward $E(\bar{\omega})$ when n tends toward an infinite value, one would certainly suspect they are, but it would be wiser to demonstrate this property more rigorously.

Let's assume that in the collected sample, everything is like randomly collecting successive fragments with average masses $\bar{\omega}_i$ and metal contents q_i, and that each one of these two quantities obeys the same probability law for all the fragments. Then, when n fragments have been collected, a sample is obtained with the following characteristics:

$$X_n=\sum_{i=1}^{n}q_i \quad [40]$$

$$Y_n=\sum_{i=1}^{n}\bar{\omega}_i \quad [41]$$

When n changes the vector (X_n,Y_n) is a stochastic process (i.e., a vectorial process with two components X_n and Y_n defined within the discrete set of positive integers n. Because of the independence of the successive fragment selections, it constitutes a Markov process with independent and stationary increments.).

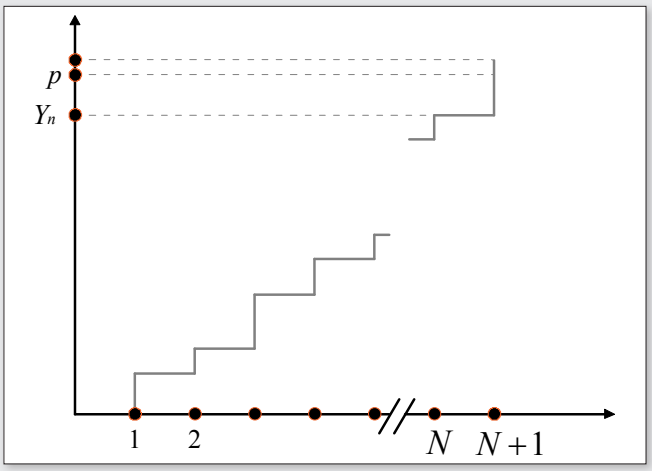

Figure 1. Illustration of the Markov process.

Each of the two components X_n and Y_n can be represented by a random steps process as illustrated in Figure 1. The sample with a number of fragments k pre-selected in advance, and studied in the former section, is defined by (X_k,Y_k), which is the value of the process (X_n,Y_n) for the particular value $n = k$.

The sample of mass p selected in advance can be defined in two different ways, either by default or by excess. As a matter of fact if N is the random time (the value of n) for which we have:

$$Y_N < p, \qquad Y_{N+1} \geq p \quad [42]$$

then it can be said that N is the random number of fragments of the sample of pre-selected mass p. Indeed, the two inequalities [42] mean that the N first fragments consist of a sample of mass smaller than p, and that the total mass of the $N + 1$ first fragments reaches or surpasses p. The sample itself can be defined either by default with characteristics X_n and Y_n or by excess with characteristics X_{n+1} and Y_{n+1}. These two definitions can be considered as equivalent; indeed both samples usually made of many fragments are different only by one fragment which is the fragment selected at a $(N + 1)^{th}$ time.

In the following developments we opted for the definition by excess (X_{N+1},Y_{N+1}). In the first step, the law of the random number of fragments N of the sample of pre-selected mass p is investigated, then in a second step, the law of the metal content of the same sample is investigated, or, in other words the law of X_{N+1}. The given mass p of the sample being assumed large, relative to the average mass $y_0 = E(\bar{\omega})$ of the individual fragments, we will be mainly searching for the asymptotic expressions of the mean and the variance of these different variables.

3.1 Law of the number of fragments *N* in the sample of pre-selected mass *p*

Let's call $P_n(p)$ the probability of having $N = n$, or, in other words, a number n *of* fragments in the sample. The event "$N = n$" coincides by definition with the event "$Y_n < P$ and $Y_{n+1} \geq p$".

Let's define $f_n(y)$ as the density of probability, and $F_n(y)$ the cumulative distribution function of the Y_n distribution. Therefore, the probability of the event "$Y_n < P$" is $F_n(p)$ and the probability of the

event "$Y_{n+1} < p$" is $F_{n+1}(p)$. Since the event "$Y_n < p$" is the logical sum of events "$N = n$" and "$Y_{n+1} < p$" that are incompatible (i.e. we have "$N = n$" OR ELSE "$Y_{n+1} < p$"), we obtain:

$$F_n(p) = P_n(p) + F_{n+1}(p) \quad [43]$$

from which the following expressions can be deduced:

$$P_n(p) = F_n(p) - F_{n+1}(p) \quad [44]$$

$$P_0(p) = 1 - F_1(p) \quad [45]$$

It is then convenient to introduce the generating function $G(s;p)$ of the $P_n(p)$ probabilities, which, according to [44] and [45] lead to:

$$G(s;p) \equiv \sum_{n=0}^{\infty} s^n P_n(p) = 1 + \sum_{n=1}^{\infty} s^{n-1}(s-1)F_n(p) \quad [46]$$

As is well known, suffices deriving the generating function and taking $s = 1$ to obtain the successive moments of the discrete law of $P_n(p)$. The two first interesting moments are:

$$E(N) = G'(1) \quad [47]$$

$$E[N(N-1)] = G''(1) \quad [48]$$

Taking into account the expression of the generating function [46] we obtain:

$$E(N) = \sum_{n=1}^{\infty} F_n(p) \quad [49]$$

$$E[N(N-1)] = 2\sum_{n=1}^{\infty}(n-1)F_n(p) \quad [50]$$

We therefore need to evaluate both sums $\Sigma F_n(p)$ and $\Sigma n F_n(p)$ when p is large. This is made possible introducing the Fourier transform and by using the following rule:

> *If* h(x) *is a function, and if* H(u) *is its Fourier transform (generally taken in its distributional expression), it is known that the continuity properties of* H(u) *give an image of the regularity of* h(x) *toward the infinite. In particular, if the distribution of* H(u) *is identified with a continuous function growing slowly,* h(x) *tends toward zero when* x *tends toward the infinite.*

Then, let $h(x)$ be a function worth zero for $x < 0$, and $H(u)$ its Fourier transform, which generally is a distribution. If the distribution

$$H(u) + \frac{a_0}{iu} - \frac{a_1}{(iu)^2} + \frac{a_2}{(iu)^3}$$

is identified with a continuous function growing slowly, then:

$$\lim\left[h(x) - a_0 - a_i x - \frac{a_2}{2}x^2\right] = 0 \quad \text{when} \quad x \to +\infty \quad [51]$$

In other words $h(x)$ is then asymptotically equal to the polynomial function

$$a_0 + a_1 x + \frac{a_2}{2}x^2 .$$

In what follows let's make $\lambda = -iu$ which is formally equivalent to using the Laplace transform. We should ignore some mathematical difficulties that are of no consequences in the following study, especially the summation of geometric series of the type $\Sigma[\Phi(u)]^n$, where $\Phi(u)$ is a characteristic function: in fact it is necessary to assume the inequality $|\Phi(u)| < 1$ is strict as soon as u is not nil. This condition is indeed verified for all usual laws, with the exception of discrete laws such as the Poisson law, for which the random variable cannot admit other values than integer multiples of a same quantity: these laws have characteristic functions that are periodic and the equality $\Phi(u = 1)$ is indeed possible for $u \neq 0$.

To find the asymptotic expression:

$$h(x) \approx a_0 + a_1 x + \frac{a_2}{2}x^2 \quad [52]$$

of a function $h(x)$ (identically nil for $x < 0$), suffice taking its Laplace transform $\Phi(\lambda)$, and then determining constants a_0, a_1 and a_2 in such a way that

$$\Phi(\lambda) - \frac{a_0}{\lambda} - \frac{a_1}{\lambda^2} - \frac{a_2}{\lambda^3}$$

is a continuous function in $\lambda = 0$.

Calculation of E(N) *and* E(N^2)

Then, let $F_1(y)$ be the function representing Y_1 (i.e. the mass of a fragment) and $\Phi(\mu)$ its Laplace transform:

$$\Phi(\mu) = \int_0^{\infty} e^{-\mu y} dF_1(y) \quad [53]$$

The variable Y_n which is the sum of n independent variables from the distribution law F_i, follows a law for which the Laplace transform is $[\Phi(\mu)]^n$. The transform of $F_n(p)$ is then $(1/\mu)[\Phi(\mu)]^n$, and the sum

$$\sum_{n=0}^{\infty} F_n(p)$$

has the following transform:

$$\frac{1}{\mu}\sum_{n=0}^{\infty}[\Phi(\mu)]^n = \frac{1}{\mu[1-\Phi(\mu)]} \quad [54]$$

[T.N.: by summation of the series].

Let $y_0 = E(Y_1)$ and σ_y^2 be the mean and the variance of the fragment mass.

[T.N.: Replacing $e^{-\mu y}$ *in [53] by its development in series* $e^{-\mu y} = 1 + (-\mu y)/1! + (-\mu y)^2/2! = \ldots$ *and integrating]* we obtain the limited development:

$$\Phi(\mu) = 1 - y_0\mu + \frac{1}{2}\mu^2(y_0^2 + \sigma_y^2) + \ldots \quad [55]$$

from which we derive:

$$\frac{1}{\mu[1-\Phi(\mu)]} = \frac{1}{y_0\mu^2} + \frac{1}{2\mu}\left(1 + \frac{\sigma_y^2}{y_0^2}\right) + \ldots \quad [56]$$

Applying the rule described above, the following asymptotic expression is deduced:

$$\sum_{n=0}^{\infty} F_n(p) \approx \frac{p}{y_0} + \frac{1}{2}\left(1 + \frac{\sigma_y^2}{y_0^2}\right) \quad [57]$$

Go back to [49] and [50] remembering that $F_0(p) = 1$. The mathematical expectation $E(N;p)$ of the number of fragments in the sample of mass p, then admits the following asymptotic expression:

$$E(N;p)=\frac{p}{y_0}+\frac{1}{2}\left(\frac{\sigma_y^2}{y_0^2}-1\right) \quad [58]$$

Its principal part coincides, as expected, with the ratio of the selected sample mass p to the average mass y_0 of individual fragments. *(T.N.: but it is not equal to it exactly, and there is a first, small bias. To understand why this surprising bias exists, one needs to go back to the definition of the sample of mass* p*, i.e. to formula [42].)*

Now, let's go to the sum $\Sigma nF_n(p)$ for which the Laplace transform relative to p is:

$$\frac{1}{\mu}\sum_{n=1}^{\infty} n\left[\Phi(\mu)\right]^n=\frac{1}{\mu}\cdot\frac{\Phi(\mu)}{\left[1-\Phi(\mu)\right]^2} \quad [59]$$

Taking the limited development, now pushed to the third order, and calling a_3 the moment of order 3 of the law Φ (i.e. F_1), we obtain:

$$\Phi(\mu)=1-y_0\mu+\frac{1}{2!}\mu^2\left[y_0^2+\sigma_y^2\right]-\frac{1}{3!}a_3\mu^3+\ldots \quad [60]$$

From which we easily obtain:

$$\frac{1}{\mu}\cdot\frac{\Phi(\mu)}{\left[1-\Phi(\mu)\right]^2}=\frac{1}{y_0^2\mu^3}+\frac{1}{\mu^2}\cdot\frac{\sigma_y^2}{y_0^3}+\frac{1}{\mu}\left[\frac{1}{4}+\frac{\sigma_y^2}{y_0^2}+\frac{3}{4}\cdot\frac{\sigma_y^4}{y_0^4}-\frac{1}{3}\cdot\frac{a_3}{y_0^3}\right]+\ldots \quad [61]$$

By applying the rule used earlier we obtain the following asymptotic expression:

$$\sum_{n=1}^{\infty} nF_n(p)\approx\frac{1}{2}\cdot\frac{p^2}{y_0^2}+p\cdot\frac{\sigma_y^2}{y_0^3}+\frac{1}{4}+\frac{\sigma_y^2}{y_0^2}+\frac{3}{4}\cdot\frac{\sigma_y^4}{y_0^4}-\frac{1}{3}\cdot\frac{a_3}{y_0^3} \quad [62]$$

In fact (for p large) we only need the terms in p and p^2 and we can ignore the constant term. Transposing this result into [49] and [50], and taking into account expression [58] for $E(N)$, we obtain:

$$E\left(N^2\right)=\frac{p^2}{y_0^2}+\left(2\cdot\frac{\sigma_y^2}{y_0^2}-1\right)\frac{p}{y_0} \quad [63]$$

Then by elevating [58] to a square and subtracting it from the above expression [63], we finally find the asymptotic expression of the variance of the number of fragments in the sample of mass p:

$$D^2(N;p)=\frac{p}{y_0}\cdot\frac{\sigma_y^2}{y_0^2} \quad [64]$$

This expression is proportional to p/y_0 therefore proportional to $E(N;p)$ at the first order.

3.2 Law for the metal content X_{N+1} of the sample with a given mass

Now, let's call $f(x,y)$ the density of probability of the (X_n,Y_n) characteristics of a single fragment, and let $\Phi(\lambda,\mu)$ be its Laplace transform:

$$\Phi(\lambda,\mu)=\int_0^{\infty}\int_0^{\infty} e^{-\lambda x-\mu y}f(x,y)\,dxdy \quad [65]$$

We must determine the density $g(x;p)$ of the metal content X_{N+1} of the sample (by excess) of a given mass p and random number of fragments N.

[T.N.: $g(x,p)=\int_0^{\infty} f(x,y)\,dy$*]*

To express this law, and in particular to find the asymptotic expressions of the mean and variance of X_{N+1} we shall use the Laplace transform $\Gamma(\lambda,\mu)$ of the function $g(x;p)$, relative to the two variables x and p:

$$\Gamma(\lambda,\mu)=\int_0^{\infty}\int_0^{\infty} g(x;p)\,e^{-\lambda x-\mu p}dxdp \quad [66]$$

We will take advantage of the fact that the process (X_n,Y_n) is made of stationary and independent increments, and more precisely that for all $n>1$, the vector (X_n-X_1,Y_n-Y_1) is independent of (X_1,Y_1) and follows the same probability law that of (X_{n-1},Y_{n-1}). The sample of a given mass p has a number of fragments $N=0$ if the first fragment has a mass $Y_1\geq p$. On the contrary if the first fragment has a mass $Y_1=\eta<p$ and a metal content $X_1=\varepsilon$ the conditional probability law of the sample of given mass p (tied by conditions $X_1=\varepsilon$ and $Y_1=\eta$) has a density $g(x-\varepsilon;p-\eta)$. We then deduce the integral equation:

$$g(x,p)=\int_p^{\infty} f(x,y)\,dy+\int_0^{x} d\varepsilon\int_0^{p} g(x-\varepsilon;p-\eta)\,f(\varepsilon,\eta)\,d\eta \quad [67]$$

If we apply the Laplace Transform (in x and p), to both members of equation [67], the convolution products they contain are replaced by ordinary multiplicative products and we obtain:

$$\Gamma(\lambda,\mu)=\frac{1}{\mu}\left[\Phi(\lambda,0)-\Phi(\lambda,\mu)\right]+\Gamma(\lambda,\mu)\,\Phi(\lambda,\mu) \quad [68]$$

From [68] we immediately deduce the expression of the Laplace transform $\Gamma(\lambda,\mu)$ of the density $g(x;p)$:

$$\Gamma(\lambda,\mu)=\frac{1}{\mu}\cdot\frac{\Phi(\lambda,0)-\Phi(\lambda,\mu)}{1-\Phi(\lambda,\mu)} \quad [69]$$

By deriving this transform in λ and by making $\lambda=0$ we obtain the Laplace transforms for the mathematical expectation $E(X;p)$ and the variance $E(X^2;p)$, which are functions of the only variable p. To write this in a concise way we will call the metal content of the sample by excess X, instead of calling it X_{N+1}. By using the Laplace transforms, the rule already used earlier will then allow us to calculate the asymptotic expressions representing these two mathematical expectations, and, as a result, the variance of the sample with a given mass p.

Calculation of E(X;p)

By deriving [69] once in λ and by making $\lambda=0$ we obtain the transform of $E(X;p)$ under the following form:

$$-\frac{1}{\mu}\cdot\frac{\Phi'\lambda(0,0)}{1-\Phi(0,\mu)}=\frac{x_0}{\mu}\sum_{n=0}^{\infty}\left[\Phi(0,\mu)\right]^n \quad [70]$$

where designates the mathematical expectation $E(X_1)$ of the metal content of a given fragment. In the same way σ_x^2 will designate the variance of X_1.

Since $(1/\mu)[\Phi(0,\mu)]^n$ is the transform of $F_n(p)$ according to [49] and [50] we obtain:

$$E(X;p)=x_0\left[1+E(N;p)\right] \quad [71]$$

Then, the by-excess sample, with a number of fragments $N+1$ instead of N, contains, on average, a metal content proportional to its size. As far as the mathematical expectations are concerned, the condition that we imposed to the sample by fixing its mass p does

not generate any effect: everything remains as if it was the collection of a given number of fragments $n = 1 + E(N;p)$.

Taking into account [58] we also obtain:

$$E(X;p) = \frac{x_0}{y_0}p + \frac{1}{2}x_0\left(1+\frac{\sigma_y^2}{y_0^2}\right) \quad [72]$$

The mathematical expectation $(1/p)E(X;p)$ of the metal concentration X/p of the sample of given mass p is different than the real concentration x_0/y_0 by a quantity that is always positive, but very small (in x_0/p).

This tiny, positive bias is easily explained if we recall that we are dealing with a sample by excess with a real mass Y_{N+1} that is always slightly superior to p (see [42]).

Calculation of $E(X^2;p)$

We obtain the Laplace transform of the order 2 moment by deriving $\Gamma(\lambda,\mu)$ twice in λ before making $\lambda = 0$, which according to [69] gives:

$$\frac{1}{\mu}\cdot\frac{\Phi_\lambda^{''}2(0,0)}{1-\Phi(0,\mu)} + \frac{2}{\mu}\cdot\frac{\Phi_\lambda^{'}(0,0)\Phi_\lambda^{'}(0,\mu)}{\left[1-\Phi(0,\mu)\right]^2}$$
$$= \frac{1}{\mu}\cdot\frac{x_0^2+\sigma_x^2}{1-\Phi(0,\mu)} - \frac{2x_0}{\mu}\cdot\frac{\Phi_\lambda^{'}(0,\mu)}{\left[1-\Phi(0,\mu)\right]^2} \quad [73]$$

The first term is different from the one that we used to calculate $E(X;p)$ only by a constant factor. Then, it corresponds to a term with a principal part in p in expression $E(X^2;p)$, which is $(x_0^2 + \sigma_x^2)(p/y_0)$.

Developing the second term, we easily obtain:

$$-\frac{2x_0}{\mu}\cdot\frac{\Phi_\lambda^{'}(0,\mu)}{\left[1-\Phi(0,\mu)\right]^2} = \frac{2x_0^2}{\mu^3 y_0^2}\left[1+\mu\left(\frac{\sigma_y^2}{y_0}-\frac{\sigma_{xy}}{x_0}\right)+\ldots\right] \quad [74]$$

The corresponding asymptotic expression is then (limiting ourselves to the first two main terms in p and p^2):

$$\frac{x_0^2}{y_0^2}p^2 + 2\frac{x_0^2}{y_0^2}\left(\frac{\sigma_y^2}{y_0}-\frac{\sigma_{xy}}{x_0}\right)p$$

By grouping these two results together we finally obtain:

$$E(X^2;p) = \frac{x_0^2}{y_0^2}p^2 + x_0^2\frac{p}{y_0}\left(1+\frac{\sigma_x^2}{x_0^2}+2\frac{\sigma_y^2}{y_0^2}-2\frac{\sigma_{xy}}{x_0y_0}\right) \quad [75]$$

Calculating the variance of the sample of given mass p

It is only necessary to square equation [72] to make this calculation, of course limiting the calculation to the terms p and p^2, and then subtracting it from equation [75] which leads to the principal part of the variance:

$$D^2(X;p) = p\frac{x_0^2}{y_0}\left[\frac{\sigma_x^2}{x_0^2}+\frac{\sigma_y^2}{y_0^2}-2\frac{\sigma_{xy}}{x_0y_0}\right] \quad [76]$$

The expression between brackets can be replaced by:

$$E\left[\left(\frac{X_1}{x_0}-\frac{Y_1}{y_0}\right)^2\right] = E\left[\left(\frac{q}{x_0}-\frac{\bar{\omega}}{y_0}\right)^2\right] \quad [77]$$

since $Y_1 = \bar{\omega}$ is the average mass of one elementary fragment and $X_1 = q$ is the metal content of that fragment. Therefore the concentration X/p of the sample of a given mass p has a variance equal to:

$$\frac{1}{p^2}D^2(X;p) = \frac{y_0}{p}\cdot\frac{x_0^2}{y_0^2}E\left[\left(\frac{q}{x_0}-\frac{\bar{\omega}}{y_0}\right)^2\right] \quad [78]$$

Finally, in a first order in $1/p$ according to [58] we can replace y_0/p by $1/E(N;p)$ and obtain:

$$\frac{1}{p^2}D^2(X;p) = \frac{1}{E(N;p)}\left(\frac{x_0}{y_0}\right)^2 E\left[\left(\frac{q}{x_0}-\frac{\bar{\omega}}{y_0}\right)^2\right] \quad [79]$$

Now, let's go back to equation [39]. We find that the sample of a given mass p has the same variance as the sample with a given number of fragments n, on the condition, of course, that we select for n the mathematical expectation $E(N;p)$ of the random number of fragments in the sample of a given mass p. This result is valid if p or $E(N)$ are large enough in order to ignore the terms in $1/p^2$ or $1/[E(N)]^2$.

4. Conclusion

We finally reach the point where the full justification of the calculation mode selected by Pierre Gy has been achieved once and for all. As far as we are concerned, we are satisfied to have, through this study, so arid for many, brought our own contribution to this fundamental piece of work, so critically important for the foundation of the Theory of Sampling. Also, and but not the least, this work has allowed us to refute without appeal the many criticisms unfairly made over the years to Pierre Gy's work.

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Recommended reading

1. Gy, P.M., "L'Echantillonage des Minerais en Vrac (Sampling of particulate materials). Volume 1", *Revue de l'Industrie Minerale*, St. Etienne, France. Numero Special (Special issue, January 15, 1967).
2. Gy, P.M., "L'Echantillonage des Minerais en Vrac (Sampling of particulate materials). Volume 2", *Revue de l'Industrie Minerale*, St. Etienne, France. Numero Special (Special issue, September 15, 1971).