doi: 10.1255/tosf.79

Introduction and first ever rigorous derivation of the liberation factor

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A simplified approach to the demonstration of Gy's Theory of Sampling (TOS) sampling variance formula is proposed, with the added advantage that it clarifies the real assumptions that are necessary to lead the demonstration to its end. In the process, the introduction of the liberation factor in TOS is also clarified and, for the first time, a rigorous definition of that factor is offered, which naturally lead to its modelling in past years using geostatistical concepts. A generalised from for the mineralogical factor c is also proposed in an appendix.

Introduction

We will use the following notations and conventions:

Large lot L: Mass = M_L, made of N_L fragments $f_j(t_j, m_j)$ where t_j and m_j are the grades and masses of individual fragments
Sample S: Mass = M_S, made of N_S fragments $f_i(t_i, m_i)$

For the lot (summations on sub-index j by convention):

$$\bar{m}_L = M_L / N_L = \Sigma_j\, m_j / N_L$$

$$t_L = \Sigma_j\, m_j\, t_j / M_L = \Sigma_j\, (m_j\, t_j / \bar{m}_L) / N_L \quad \text{(grade of the lot)}$$

Similarly for a small sample (summations on sub-index i by convention):

$$\bar{m}_S = M_S / N_S = \Sigma_i\, m_i / N_S$$

$$t_S = \Sigma_i\, m_i\, t_i / M_S = \Sigma_i\, (m_i\, t_i / \bar{m}_S) / N_S$$

Important assumptions

We will assume sampling in number (the equivalence to sampling in mass was established by Matheron[1] (2015) so that N_S is a fixed number for all the samples.

$\mathrm{Var}(m_i)$ is limited and N_S is a very large number, so that $\mathrm{Var}(\bar{m}_S) = \mathrm{Var}(m_i) / N_S$ is small. Additionally, if the sample is correct, $E(\bar{m}_S) = \bar{m}_L$. As a result:

$\bar{m}_S$ can be assimilated to $\bar{m}_L$ in "good approximation"

This FUNDAMENTAL approximation is equivalent to assuming exact representation in the sample of the average fragment mass in the lot. This, which includes the neglecting of the small variations in total mass M_S from sample to sample, is the origin of the mathematical difficulty of Gy's and Matheron's rigorous demonstrations, and of the first order approximation that characterises their result.

Relative sampling error

Under these conditions, the relative sampling error is:

$$\varepsilon_{RS} = (t_S - t_L) / t_L = \Sigma_i\, [(m_i / \bar{m}_L)(t_i - t_L) / t_L]\, N_S = \Sigma_i\, h_i / N_S$$
(i.e. the *arithmetic* mean of the independent h_i in S)

where $h_i = [(t_i - t_L) / t_L](m_i / \bar{m}_L)$ (see TOS terminology in Appendix 1)
Properties of h_i:

$$\bar{h}_L = \Sigma_j\, (m_j\, t_j / \bar{m}_L) / (t_L\, N_L) - t_L / t_L = 1 - 1 = 0, \text{ therefore}$$

$$\mathrm{Var}(h_i) = \Sigma_j\, [(t_j - t_L) / t_L]^2 (m_j / \bar{m}_L)^2 / N_L = CH_L$$
(see TOS terminology in Appendix 1)

Relative sampling variance

Since: $\bar{\varepsilon}_{RS} = 0$

$$\sigma_R^2 = \mathrm{Var}(\varepsilon_{RS}) = \mathrm{Var}(h_i) / N_S = CH_L / N_S = (\bar{m}_L / M_S)\, CH_L$$

And as: $CH_L = \Sigma_j\, [(t_j - t_L) / t_L]^2\, m_j^2 / (\bar{m}_L\, M_L)$:

$$\sigma_R^2 = (1 / M_S)\, \Sigma_j\, [(t_j - t_L) / t_L]^2\, m_j^2 / M_L \quad \text{(for large lots)}$$

For smaller lots, because (in geostatistical notations):

$$D^2(M_S \mid \infty) = D^2(M_S \mid M_L) + D^2(M_L \mid \infty)$$

in all cases, we have:

$$\sigma_R^2 = (1 / M_S - 1 / M_L)\, \Sigma_j\, [(t_j - t_L) / t_L]^2\, m_j^2 / M_L$$

This is Gy's and Matheron's first order approximation of the sampling variance.

Below liberation, by separating mineral and gangue fragments in the summation, this formula can easily be transformed into the fully calculable quantity (see demonstration in Appendix 2):

$$\sigma_R^2 = (1 / M_S - 1 / M_L)\, c\, f_G\, g\, d_{NG}^3 = (1 / M_S - 1 / M_L)\, C\, d_N^3$$

where C is a calculable constant in which **c is a *generalized* mineralogical constant**, and $d_{NG} = d_N$ is the comminution size of the lot.

Above liberation, it is not calculable, but somewhat smaller than if liberation had been achieved, so, introducing a number ℓ between 0 and 1, we can write:

$$\boldsymbol{\sigma_R^2 = (1 / M_S - 1 / M_L)\, c\, f_G\, \ell\, d_N^3 \quad \text{with } 0 < \ell < 1}$$

Gy's well-known theory ends here, with no status given to ℓ, unfortunately precluding its (necessary) modelling.

We need to go further and:

- uncover the physical meaning of ℓ
- find the factors affecting it
- find a model for its variations

Important note:
One should mention there was one valuable attempt by Gy at modelling ℓ using the maximum grade achievable by a fragment at nominal size (Pitard, 2015).[2] This work was of theoretical interest, but it was not practical and only established (therefore valid) under very restrictive conditions.

However, in Francis Pitard's words, this method *"...showed quite well the relation of the liberation factor with mineralogy. For example, the liberation curve may look completely different if it is individual and isolated gold particle that liberate, or if it is a cluster made of many particles side by side, or if you prefer an aggregate of many particles."*

This said, full, general modelling of ℓ calls for geostatistical concepts that were simply not available at the time Gy was publishing TOS.

True definition of the liberation factor

In the liberated case, for a given, correct sample S, from a much larger, liberated lot:

$$\text{Rel.Var.}[S] = c\ f\ g\ d_N^3 / M_S$$

As we stated above, reasoning shows that when the ore is NOT liberated, the sampling variance is necessarily somewhat lower. As a result, for a given, sample S, from a large, non-liberated lot, by introducing a number ℓ between 0 and 1, Gy wrote:

$$\text{Rel.Var.}[S] = [c\ f\ g\ d_N^3 / M_S]\ \ell \quad \text{with } \mathbf{0 < \ell < 1} \tag{1}$$

Now, let S_{Lib} be a correct sample, taken the same way, in the LIBERATED lot, with the same average number of fragments N as in S, i.e. with mass M_{Lib} such that:

$$M_{Lib} = N\ \bar{m}_{Lib} \quad \text{with } N = M_S / \bar{m}_L$$

$\bar{m}_{Lib}$ and $\bar{m}_L$ being the average fragment masses in the liberated and non-liberated lots.

$$\text{Rel.Var.}[S_{Lib}] = c\ f\ g\ d_\ell^3 / M_{Lib} \tag{2}$$

In our model:

$$\bar{m}_{Lib} / \bar{m}_L = f\ g\ d_\ell^3 / f\ g\ d_N^3 = (d_\ell / d_N)^3 = M_{Lib} / M_S$$

So that we can replace M_{Lib} by $M_S\ (d_\ell / d_N)^3$ in (2):

$$\text{Rel.Var.}[S_{Lib}] = c\ f\ g\ d_\ell^3 / M_{Lib} = c\ f\ g\ d_\ell^3 / [M_S\ (d_\ell / d_N)^3]$$

Or:

$$\text{Rel.Var.}[S_{Lib}] = c\ f\ g\ d_N^3 / M_S \tag{3}$$

Therefore, dividing (1) by (3):

$$\boldsymbol{\ell = \text{Rel. Grade Var.}[S] / \text{Rel. Grade Var.}[S_{Lib}]} \tag{4}$$

This new equation, *ratio of the sample variance to the variance of the liberated sample with the same average number of fragments*, is valid and constant for any sample (or sample mass), and provides us with a precise, rigorous and *objective* definition of factor ℓ.

Models for ℓ based on this characterisation have been the objects of numerous papers by the author. Equation (4) amounts to a ratio of variances of two different fragment sizes (in the lot being sampled and in the liberated one), and this ratio could only be modeled by drawing from geostatistical considerations.

References

1. G. Matheron, "Comparison between samples with constant mass and samples with constant fragment population size (and calculations of their sampling variances)", Translated from French to English, clarified and further commented by D. François-Bongarçon and F. Pitard, in *Proceedings of the 7th International Conference on Sampling and Blending*, Ed by K.H. Esbensen and C. Wagner, *TOS forum* **Issue 5,** in press (2015). doi: http://dx.doi.org/10.1255/tosf.80
2. F.F. Pitard, *Pierre Gy's Sampling Theory and Sampling Practice*, 2nd Edn. CRC Press (1993). ISBN 0-8493-8917-8

Appendix 1

Some TOS Classical Definitions

RELATIVE HETEROGENEITY carried by fragment f_i in lot L:

$$h_i = [(t_i - t_L) / t_L]\ (m_i / \bar{m}_L)$$

CONSTITUTION HETEROGENEITY of lot L:

$$\begin{aligned} CH_L &= \Sigma_j\ [(t_j - t_L) / t_L]^2\ (m_j / \bar{m}_L)^2\ (\bar{m}_L / M_L) \\ &= \Sigma_j\ [(t_j - t_L) / t_L]^2\ m_j^2 / (\bar{m}_L\ M_L) \end{aligned}$$

It is a characteristic (weighted variance) of the "average fragment" in the lot and it measures the intrinsic variability of the lot.

HETEROGENEITY INVARIANT of lot L:

$$H_L = CH_L\ \bar{m}_L$$

Appendix 2
At liberation size and below

$$\sigma_R^2 = (1 / M_S - 1 / M_L)\ c\ f_G\ g\ d_{NG}^3 = (1 / M_S - 1 / M_L)\ C\ d_N^3$$

Summing on Mineral (M) and Gangue (G) with respective sub-indices i and j:

$$\sigma_R^2 = (1 / M_S - 1 / M_L)\{{}^{M}\Sigma_j\ [(t_j - t_L) / t_L]^2\ m_j^2 / M_L + {}^{G}\Sigma_i\ [(t_i - t_L) / t_L]^2\ m_i^2 / M_L\}$$

For the mineral, $t_j = 1$ and for gangue $t_i = 0$, so, introducing the densities ρ_M and ρ_G and fragment volumes v_i and v_j:

$$\sigma_R^2 = (1 / M_S - 1 / M_L)\ \{[(1 - t_L) / t_L]^2\ {}^{M}\Sigma_j\ m_j^2 / M_L + {}^{G}\Sigma_i\ m_i^2 / M_L\}$$

$$\sigma_R^2 = (1 / M_S - 1 / M_L)\ \{[(1 - t_L) / t_L]^2\ \rho_M\ {}^{M}\Sigma_j\ v_j\ m_j / M_L + \rho_G\ {}^{G}\Sigma_i\ v_i\ m_i / M_L\}$$

Now:

$${}^{M}\Sigma_j\ v_j\ m_j / M_L = (M_M / M_L)\ {}^{M}\Sigma_j\ [(m_j / M_M)\ v_j]$$

$M_M / M_L = t_L$ and ${}^{M}\Sigma_j\ [(m_j / M_L)\ v_j]$ is the mass-weighted average mineral fragment volume $\bar{v}_M$ in the lot, so that:

$${}^{M}\Sigma_j\ v_j\ m_j / M_L = t_L\ \bar{v}_M$$

Similarly:

$${}^{G}\Sigma_i\ v_i\ m_i / M_L = (1 - t_L)\ \bar{v}_G$$

where $\bar{v}_G$ is the mass-weighted average gangue fragment volume in the lot.
So now:

$$\sigma_R^2 = (1 / M_S - 1 / M_L)\ \{[(1 - t_L) / t_L]^2\ \rho_M\ t_L\ \bar{v}_M + \rho_G\ (1 - t_L)\ \bar{v}_G\}$$

finally:

$$\sigma_R^2 = (1 / M_S - 1 / M_L)\ [(1 - t_L) / t_L]\ [(1 - t_L)\ \rho_M\ \bar{v}_M + t_L\ \rho_G\ \bar{v}_G]$$

Let us introduce the volume ratio $k = \bar{v}_M / \bar{v}_G$:

$$\sigma_R^2 = (1 / M_S - 1 / M_L)\ [(1 - t_L) / t_L]\ [(1 - t_L)\ k\ \rho_M + t_L\ \rho_G]\ \bar{v}_G$$

In this expression,

$$\mathbf{c = [(1 - t_L) / t_L]\ [(1 - t_L)\ k\ \rho_M + t_L\ \rho_G]}$$

is the "**generalised** mineralogical factor" and

$$\sigma_R^2 = (1 / M_S - 1 / M_L)\ c\ \bar{v}_G$$

This is a very important formula, which actually is the **true** variance formula. It shows the lot behaves, in terms of sampling it, exactly as a hypothetical lot with all fragments having the same size corresponding to the average gangue fragment volume.

Calculation of constant k in factor c

Introducing Gy's classical granulometric factor g, liberation size d_ℓ, the shape factors f_M and f_G and the nominal comminution sizes $d_{NM}\ \sigma\ d_\ell$ and $d_{NG}\ \sigma\ d_\ell$ of mineral and gangue:

$$\bar{v}_M = f_M\ g\ d_{NM}^3 \quad \text{and} \quad \bar{v}_G = f_G\ g\ d_{NG}^3$$

$$k = \bar{v}_M / \bar{v}_G = (f_M / f_G)\ (d_{NM} / d_{NG})^3$$

Particular cases:

- Mineral and gangue comminute together and have the same shape factors:
  then $k = 1$
- Mineral and gangue comminute together and have different shape factors:
  then $k = f_M / f_G$
- Mineral does not comminute below liberation size (e.g. gold grains) while gangue is comminuted to size d_N:
  then: $k = (f_M / f_G)\,(d_\ell / d_N)^3$ in general,
  or: $k = (d_\ell / d_N)^3$ if $f_M = f_G$
- Special cases can be calculated as well. For instance, if the mineral has a unique size instead of a size distribution, one can take $g = 1$ for mineral grains in the definition of $\bar{v}_M$.

In all cases, the relative variance then becomes:

$$\sigma_R^2 = (1 / M_S - 1 / M_L)\, c\, f_G\, g\, d_{NG}^3 = (1 / M_S - 1 / M_L)\, C\, d_{NG}^3$$

where C is a calculable constant.

Note: in the case of non-comminutable gold (third bullet above), this expression reduces to the following approximation:

$$\sigma_R^2 = (1 / M_S - 1 / M_L)\,(\rho_M / t_L)\, f_M\, g\, d_N^3$$

which is why, in that case, the shape factor to be used is that of the gold grains.